

# “Gauge” in General Relativity :

— Second-order general relativistic gauge-invariant perturbation theory —

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## Abstract

As in the case of the other gauge field theories, there is so called “gauge” also in general relativity. This “gauge” is unphysical degree of freedom. There are two kinds of “gauges” in general relativity. These are called the first- and the second-kind of gauges, respectively. The gauge of the first kind is just coordinate system on a single manifold. On the other hand, the gauge of the second kind arises in the general relativistic perturbations. Through the precise distinction of these two concepts of “gauges”, we develop second-order gauge-invariant general relativistic perturbation theory.

## 1 Introduction

General relativity is regarded as a gauge theory. In 1956, Uchiyama pointed out that gravitational field is introduced by the invariance of the action under local Lorentz transformations [1]. This is due to general covariance in general relativity. Since this local Lorentz group is a group of coordinate transformations, coordinate system is called “gauge” in general relativity from this history.

General covariance intuitively states that there is no preferred coordinate system in nature and it also introduce “gauge” in the theory. This “gauge” is the unphysical degree of freedom and we have to fix the “gauge” or to extract some invariant quantities to obtain physical results. Thus, treatments of “gauge” are crucial in general relativity. This situation becomes more delicate in general relativistic perturbation theory.

On the other hand, the general relativistic higher-order perturbation theory is a topical subject in recent physics, for example, cosmological perturbations, perturbations of a black hole and a star. So, it is necessary to formulate the higher-order general relativistic perturbation theory from general point of view.

In this article, we clarify the notion of “gauges” in general relativity, which is necessary to develop general relativistic higher-order “gauge-invariant” perturbation theory. The details of this perturbation theory can be seen in Ref. [2].

## 2 “Gauge” in general relativity

In 1964, Sachs [3] pointed out that there are two kinds of “gauges” in general relativity which are closely related to the general covariance. He called these two “gauges” as the first- and the second-kind of gauges, respectively.

*The first kind gauge* is a coordinate system on a single manifold  $\mathcal{M}$ . On a manifold, we can always introduce a coordinate system as a diffeomorphism  $\psi_\alpha$  from an open set  $O_\alpha \subset \mathcal{M}$  to an open set  $\psi_\alpha(O_\alpha) \subset \mathbb{R}^n$  ( $n = \dim \mathcal{M}$ ). This diffeomorphism  $\psi_\alpha$ , i.e., coordinate system of the open set  $O_\alpha$  is called *gauge choice* (of the first kind). If we consider another open set in  $O_\beta \subset \mathcal{M}$ , we have another gauge choice  $\psi_\beta : O_\beta \mapsto \psi_\beta(O_\beta) \subset \mathbb{R}^n$  for  $O_\beta$ . The diffeomorphism  $\psi_\beta \circ \psi_\alpha^{-1}$  is called *gauge transformation* (of the first kind), which is a coordinate transformation :  $\psi_\alpha(O_\alpha \cap O_\beta) \subset \mathbb{R}^n \mapsto \psi_\beta(O_\alpha \cap O_\beta) \subset \mathbb{R}^n$ .

According to the theory of a manifold, coordinate systems are not on a manifold itself but we can always introduce the coordinate system through a map from an open set in the manifold  $\mathcal{M}$  to an open set of  $\mathbb{R}^n$ . For this reason, general covariance in general relativity is automatically included in the premise that our spacetime is regarded as a single manifold. The first kind gauge arises due to this general covariance but it is harmless if we apply the covariant theory on the manifold in many cases.

*The second kind gauge* appears in general relativistic perturbations. To explain this, we have to remind what we are doing in perturbation theory. First, in any perturbation theories, we always treat two spacetime manifolds. One is the physical spacetime  $\mathcal{M}$ , which we want to describe by perturbations, and the other is the background spacetime  $\mathcal{M}_0$ , which is prepared for perturbative analyses by us. Note that these two spacetimes  $\mathcal{M}$  and  $\mathcal{M}_0$  are distinct. Second, in any perturbation theories, we always write equations in the form

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p) \quad (1)$$

as the perturbation of the variable  $Q$ . Keeping in our mind that we always treat two different spacetimes,  $\mathcal{M}$  and  $\mathcal{M}_0$ , in perturbation theory, Eq. (1) is a rather curious equation because the variable on the left-hand side of Eq. (1) is a variable on the physical spacetime  $\mathcal{M}$ , while the variables on the right-hand side of Eq. (1) are variables on the background spacetime,  $\mathcal{M}_0$ . In short, Eq. (1) gives a relation between variables on two different manifolds.

We note that, through Eq. (1), we have implicitly identified points in these two different manifolds. More specifically, the point “ $p$ ” in  $Q(\text{“}p\text{”})$  on the left-hand side of Eq. (1) is on  $\mathcal{M}$ . Similarly, the point  $p$  in  $Q_0(p)$  or  $\delta Q(p)$  on the right-hand side of Eq. (1) is on  $\mathcal{M}_0$ . Because Eq. (1) is regarded as an field equation, it implicitly states that the points “ $p$ ”  $\in \mathcal{M}$  and  $p \in \mathcal{M}_0$  are same. This implies that we are assuming the existence of a map  $\mathcal{M}_0 \rightarrow \mathcal{M} : p \in \mathcal{M}_0 \mapsto \text{“}p\text{”} \in \mathcal{M}$ , which is a *gauge choice* (of the second kind) [4].

It is important to note that the second kind gauge choice between  $\mathcal{M}_0$  and  $\mathcal{M}$  is not unique to the theory with general covariance. Rather, Eq. (1) involves the degree of freedom in the choice of the map  $\mathcal{X} : \mathcal{M}_0 \mapsto \mathcal{M}$ . This is called the *gauge degree of freedom* (of the second kind). This gauge degree of freedom

always exists in perturbations of a theory with general covariance. If general covariance is not imposed, there is a preferred coordinate system in the theory, and we naturally introduce this coordinate system onto both  $\mathcal{M}_0$  and  $\mathcal{M}$ . Then, we can choose the identification map  $\mathcal{X}$  using this preferred coordinate system. However, general covariance states that there is no such coordinate system, and we have no guiding principle to choose the identification map  $\mathcal{X}$ . Indeed, we may identify “ $p$ ”  $\in \mathcal{M}$  with  $q \in \mathcal{M}_0$  ( $q \neq p$ ) instead of  $p \in \mathcal{M}_0$  by the *different gauge choice*  $\mathcal{Y}$  of the second kind.

### 3 Gauge transformations and gauge invariant variables

To define the perturbation of an arbitrary tensor field  $Q$ , we consider the one-parameter family of spacetimes  $\mathcal{M}_\lambda$  so that  $\mathcal{M}_\lambda = \mathcal{M}$  and  $\mathcal{M}_{\lambda=0} = \mathcal{M}_0$ , where  $\lambda$  is an infinitesimal parameter for perturbations. We regard  $\{\mathcal{M}_\lambda | \lambda \in \mathbb{R}\}$  as an extended manifold. The gauge choice is made by assigning a diffeomorphism  $\mathcal{X}_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_\lambda$  on this extended manifold. A tensor field  $Q$  on  $\mathcal{M}_\lambda$  is pulled-back by  $\mathcal{X}_\lambda^* : Q \mapsto \mathcal{X}_\lambda^* Q$  to a tensor  $\mathcal{X}_\lambda^* Q$  on  $\mathcal{M}_0$  and we expand

$$\mathcal{X}_\lambda^* Q|_{\mathcal{M}_0} = Q_0 + \lambda^{(1)}_{\mathcal{X}} Q + \frac{1}{2} \lambda^{(2)}_{\mathcal{X}} Q + O(\lambda^3). \quad (2)$$

This defines the first- and the second-order perturbations  $^{(1)}_{\mathcal{X}} Q$  and  $^{(2)}_{\mathcal{X}} Q$  of a physical variable  $Q_\lambda$  under the gauge choice  $\mathcal{X}_\lambda$ .

When we have two different gauges  $\mathcal{X}_\lambda$  and  $\mathcal{Y}_\lambda$ , Eq. (2) defines two different representations of the  $n$ -th order perturbations  $^{(n)}_{\mathcal{X}} Q$  and  $^{(n)}_{\mathcal{Y}} Q$  on  $\mathcal{M}$ , respectively. We say that  $Q$  is *gauge invariant up to order  $n$*  iff for any two gauges  $\mathcal{X}_\lambda$  and  $\mathcal{Y}_\lambda$  the following holds:  $^{(k)}_{\mathcal{X}} Q = ^{(k)}_{\mathcal{Y}} Q$  for all  $k$  with  $k < n$ .

The *gauge transformation* is simply the change of the point identification map  $\mathcal{X}_\lambda$  to another one. If we have two different gauges  $\mathcal{X}_\lambda$  and  $\mathcal{Y}_\lambda$ , the change of the gauge choice from  $\mathcal{X}_\lambda$  to  $\mathcal{Y}_\lambda$  is represented by the diffeomorphism  $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda$ . This diffeomorphism  $\Phi_\lambda$  is the map  $\Phi_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_0$  for each value of  $\lambda \in \mathbb{R}$ . Since the diffeomorphism  $\Phi_\lambda$  does change the point identification, the diffeomorphism  $\Phi_\lambda$  is regarded as the gauge transformation  $\Phi_\lambda : \mathcal{X}_\lambda \rightarrow \mathcal{Y}_\lambda$ .

The diffeomorphism  $\Phi_\lambda$  induces a pull-back from the representation  $\mathcal{X}_\lambda^* Q_\lambda$  in the gauge  $\mathcal{X}_\lambda$  to the representation  $\mathcal{Y}_\lambda^* Q_\lambda$  in the gauge  $\mathcal{Y}_\lambda$ , i.e.,  $\mathcal{Y}_\lambda^* Q_\lambda = \Phi_\lambda^* \mathcal{X}_\lambda^* Q_\lambda$ . Further, generic arguments of the Taylor expansion of the pull-back of a tensor field on a manifold leads

$$\Phi_\lambda^* \mathcal{X}_\lambda^* Q = \mathcal{X}_\lambda^* Q + \lambda \mathcal{L}_{\xi_1} \mathcal{X}_\lambda^* Q + \frac{\lambda^2}{2} \{ \mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2 \} \mathcal{X}_\lambda^* Q + O(\lambda^3), \quad (3)$$

where  $\xi_1^a$  and  $\xi_2^a$  are the generators of the diffeomorphism  $\Phi_\lambda$ . The comparison with Eqs. (2) and (3) leads gauge transformation rule of each order:

$$^{(1)}_{\mathcal{Y}} Q - ^{(1)}_{\mathcal{X}} Q = \mathcal{L}_{\xi_1} Q_0, \quad ^{(2)}_{\mathcal{Y}} Q - ^{(2)}_{\mathcal{X}} Q = 2 \mathcal{L}_{\xi_{(1)}} ^{(1)}_{\mathcal{X}} Q + \left( \mathcal{L}_{\xi_{(2)}} + \mathcal{L}_{\xi_{(1)}}^2 \right) Q_0. \quad (4)$$

Inspecting the gauge transformation rules (4), we can define the gauge invariant variables for perturbations for arbitrary matter fields [2]. We expand the

metric on  $\mathcal{M}_\lambda$  is pulled back to  $\mathcal{X}_\lambda^* \bar{g}_{ab}$  on  $\mathcal{M}_0$  and it expanded as  $\mathcal{X}_\lambda^* \bar{g}_{ab} = g_{ab} + \lambda \mathcal{X} h_{ab} + \frac{\lambda^2}{2} \mathcal{X}^2 l_{ab} + O^3(\lambda)$ , where  $g_{ab}$  is the metric on  $\mathcal{M}_0$ . First, we assume that we already know the procedure for finding gauge invariant variables for the linear metric perturbations, i.e,  $h_{ab}$  is decomposed as  $h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab}$ , where  $\mathcal{Y} \mathcal{H}_{ab} - \mathcal{X} \mathcal{H}_{ab} = 0$ , and  $\mathcal{Y} X^a - \mathcal{X} X^a = \xi_{(1)}^a$ . This assumption is correct at least in the case of cosmological perturbations [2]. Once we accept this assumption, we can show that the second-order metric perturbation  $l_{ab}$  is decomposed as  $l_{ab} =: \mathcal{L}_{ab} + 2\mathcal{L}_X h_{ab} + (\mathcal{L}_Y - \mathcal{L}_X^2) g_{ab}$ , where  $\mathcal{Y} \mathcal{L}_{ab} - \mathcal{X} \mathcal{L}_{ab} = 0$  and  $\mathcal{Y} Y^a - \mathcal{X} Y^a = \xi_{(2)}^a + [\xi_{(1)}, X]^a$ . Furthermore, using the first- and second-order gauge variant parts,  $X^a$  and  $Y^a$ , of the metric perturbations, the gauge invariant variables for an arbitrary field  $Q$  other than the metric are given by

$$^{(1)}Q := ^{(1)}Q - \mathcal{L}_X Q_0, \quad ^{(2)}Q := ^{(2)}Q - 2\mathcal{L}_X ^{(1)}Q - (\mathcal{L}_Y - \mathcal{L}_X^2) Q_0. \quad (5)$$

These imply that any first- and second-order perturbations is always decomposed into gauge invariant and gauge variant parts as Eqs. (5), respectively.

## 4 Conclusions

We have shown the general procedure to find gauge-invariant variables in the second-order general relativistic perturbation theory through the precise treatments of “gauges”. We also showed that this general procedure is applicable to cosmological perturbations and developed the second-order cosmological perturbation theory in gauge invariant manner [2]. Due to the general covariance in general relativity, all equations are given in terms of gauge invariant variables. We are planning to apply this second-order perturbation theory to clarify the non-linear physics in Cosmic Microwave Background [5].

Besides the application to cosmology, we are also planning to apply the above general framework to black hole perturbations, perturbations of a star, and post-Minkowski description of a binary system. In conclusion, the definitions (5) of gauge invariant variables have very wide applications.

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